Towards the axiomatic systems of the third millennium in Mathematics, Logic, and Computer Science*

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Introduction

In his talk, Nelson has brilliantly presented his ideas of a mathematician who gave up what he calls "Pythagorean religion", and he shows new ways that, in his opinion, mathematicians should follow. We too are exploring, since several years, the possible future ways of mathematics, but with a different perspective (see [6-10,12-15,24,25]). Nelson, in systematically developing the formalistic perspective, gets to identify mathematics with the study of its formulae, and refuses any semantical perspective that associates numbers, sets, spaces, functions, etc. to these formulae. We aknowledge the irreplaceable rôle of formulae in elaborating and communicating mathematical ideas, as well as in comparing mathematics with other branches of knowledge. We also know that the manipulations of formulae, and in particular the deduction rules, are themselves very interesting objects of mathematics, but we do not intend to reduce the whole of mathematics to manipulation of formulae. Our attitude with respect to formulae is in some sense closer to that of an ancient navigator or explorer w.r.t. maps, who takes them as indispensable tools for navigating and for communicating the discoveries of his navigations, rather than to the attitude of the modern map collector, who takes them as valuable objects.

Nelson recalled the difficulties faced by Hilbert and Brouwer in their attempts of finding a satisfying "philosophy", where the objects commomly considered by mathematicians can be located in a "natural" way (see [3,22,23]). We do not ignore these difficulties, but we neither forget the words of Shakespeare "there are more things in heaven and earth than are dreamt of in your philosophy": we think that it does not suit to declare an object unexisting on the sole ground that we have not found a satisfying philosophy that accomodates it. In some sense, Nelson asks mathematicians to "dream less", e.g. not to assign a "too realistic" meaning to the beautiful theorems on infinite dimensional spaces of Hilbert and of Banach: on the contrary, we always recall the words of Shakespeare and hence we think that the scientist who aims to understand the real things existing in heaven and earth needs to "dream more", rather than less. We think that the greatest achievements of science are the outcome of some "wonderful dreams": complex numbers, infinitesimal calculus, Newton's

*Original title: VERSO I SISTEMI ASSIOMATICI DEL 2000 IN MATEMATICA, LOGICA E INFORMATICA mechanics, general relativity, quantum mechanics, etc.; we also think that a relevant part of the actual work of experimental physicists consists in looking for visible marks of some objects "dreamt of" by theoretical physicists.

We appreciate an elegant explicit solution to a nice mathematical problem, a simple and fast method of computing; we know that they can provide a great improvement on the mere proof of an existence theorem: but we do not believe that abandoning what Hilbert named "Cantor's paradise", peopled by more or less strange sets and by infinites of any order, may help in finding explicit solutions and efficient algorithms. Rather, we would like to imagine "Cantor's paradise" even more rich and colourful than Cantor conceived it, filled up with objects of any different kinds: sets, collections, operations, formulae, languages with their semantical interpretations, variables, categories, standard and nonstandard numbers, algorithms, propositions, predicates, classical logic and other logics, etc. Admittedly, this freedom in dreaming has a corresponding duty: to translate dreams into axioms, conjectures, theorems formulated with the maximal clarity and with the utmost exactitude (so as to make them, after all, understandable and subject to critical analysis also by scholars adopting a non-realistic, purely formalistic perspective). Not only the dialogue on these themes can go on in friendship and comprehension between "formalists" and "dreamers", but it can become a valuable source of very interesting ideas for both of them. Even better, we underline that the axioms we state below are suitable to be critically discussed, and possibly modified, enriched, improved by scholars close to any current of the philosophy of sciences. Therefore it is important that a general talk concerning mathematics, logic, computer science be formulated in much clearer way than a specialistic talk. In fact the former talk has to be understandable and subject to criticism by a great variety of interlocutors, whereas it suffices that the latter talk be understood by a restricted group of specialists. In particular, a general talk about the main basic concepts of mathematics, logic and computer science is not reserved only to a restricted number of "specialists of foundations", but it has to be accessible to all those scholars of sciences and humanities who are informed with what the ancients called philosophy, i.e. love for Wisdom.

For these reasons it seems unappropate to classify our work under the category "foundations of mathematics": we do not intend to refund mathematics on safer grounds, we are rather looking for some new tracks through the forest of mathematics, logic and computer science, giving up none of the ingenious intuitions of the scholars that marked out the first roads in this forest, which is, in our opinion, yet mainly unexplored. To be sure, we have no desire to begin the compiling of the "Bourbaki of the third millennium", a task that should inevitably become uncontrollable in dimension and length (cfr. [28]), for it should encompass mathematics, logic, and computer science (disciplines that nowdays have to be considered together, at least for their basic ideas). We simply believe that we have singled out a first simple, clear, "natural" axiomatic basis, upon which it should be possible to engraft the various branches of these disciplines, e.g. standard and nonstandard analysis, classical logic and other logics, set theories, probability calculus, categories, algorithms, languages, syntax, semantics, etc. Upon this basis it should be possible to engraft also some fundamental ideas of other disciplines of science and humanities, giving wide breath to the reflexion upon the relations among different fields of knowledge. In fact, on the one hand one has to reflect upon the most relevant applications of mathematics and upon the ideas suggested to mathematicians by comparing with various scientific, technical, human, artistic disciplines. But, on the other hand, it seems convenient to think also about the deepest reasons of the greatest results achieved by joint work of mathematicians and other scholars.

We now pass to expound, in a hopefully simple and clear way, a first axiomatic basis upon which one can engraft in a "natural" way the fundamental notions of many disciplines of science and humanities. This basis originates from various reflexions upon the main notions of mathematics, logic and computer science (see, e.g. [1,4,22,26,27,29,30]) and from many conversations with scholars of different disciplines (mathematics, physics, logic, computer science, biology, history, philosophy, economics, theology, etc.). Starting from these reflexions, it seemed appropriate to overcome the so called "set theoretic reductionism", i.e. the trend to reduce all of mathematics to set theory. On the contrary, we try and include the mathematical theories, and, if possible, also other scientific theories, within a wider framework, where a critical comparison of the fundamental ideas of different disciplines be possible. Surely, the set theories proposed by Cantor, Zermelo, Gödel, Bernays, Von Neumann and other great mathematicians of this century (see [2]) stay among the highest expressions of the human mind (comparable, e.g., with Newtonian mechanics, general relativity, quantum mechanics, Dante's 'Commedia', Michelangelo's 'Moïses', Shakespeare's tragedies, etc.). However, in order to get a better understanding of the main ideas of mathematics, logic and computer science, it seems appropriate to put them in a more general framework, ruled by the two ideas of *quality* and *relation*. In fact, these disciplines, as well as physical, chemical, biological, economical, linguistical disciplines, etc., all consider *qualitatively different objects* and study *relations among these objects*. It seems therefore appropriate to propose a short, simple system of few general *qualities* and *relations*, as a general premise to the exposition of these disciplines and to the comparison of their main ideas. These qualities and relations should constitute solid and flexible grounds upon which qualities and relations specific to each science be inserted.

Fundamental qualities and relations

We deal in this chapter with the first basic ideas concerning qualities and relations, intended as primitive notions. Notice that with this acception of primitive notion we do not intend to answer either the psychological question as to which ideas present themselves first to the mind of a child, or the historical question as to which ideas have been first considered by Mankind; we simply mean that these notions cannot be reduced to other previously introduced concepts (by means of suitable definitions).

Thus we introduce as primitive notions the idea of *"quality"*, and the idea of *"enjoying a given quality"*. We stipulate that given an object x of any kind and a quality q, when we write

$$q\,x$$

we intend to say that "x enjoys the quality q". In this chapter we introduce seven fundamental qualities: $Qqal$, $Qrel$, $Qrelb$, $Qrelt$, $Qrelq$, $Qrun$, $Qrbiun$, with the following meanings:
$Qqual\,x$ means that x is a *quality*;
$Qrel\,x$ means that x is a *relation*;

$Qrelb\,x$ means that x is a *binary relation*;
$Qrelt\,x$ means that x is a *ternary relation*;
$Qrelq\,x$ means that x is a *quaternary relation*;
$Qrun\,x$ means that x is a *univocal relation*;
$Qqual\,x$ means that x is a *biunique relation*.

These seven qualities enjoy *Qqual*, hence we can write:

Axiom 1.1 – *Qqual Qqual, Qqual Qrel, Qqual Qrelb, Qqual Qrelt, Qqual Qrelq, Qqual Qrun, Qqual Qrbiun.*

The three qualities *Qrelb*, *Qrelt*, *Qrelq* are particular cases of the more general quality *Qrel*, i.e.:

Axiom 1.2 – *An element x enjoying any of the qualities Qrelb, Qrelt, Qrelq enjoys the quality Qrel as well.*

Notice that we are not excluding that there exist relations more complex than binary, ternary or quaternary relations: we do not introduce them in this chapter simply because we shall make no use of them in this paper.

After the primitive idea of enjoying a given quality, the second most important primitive idea of this section is that of *"being in a given relation"*. Namely, given two objects x, y of any kind, and a binary relation r, we write

$$r\,x, y$$

or sometimes

$$r\,x; y$$

to mean that "x and y are in the relation r". At times, instead of saying that x and y are in the relation r, we also say that x is in the relation r with y.

Similarly, if x, y, z are objects of any kind and ρ is a ternary relation, we write

$$\rho\,x, y, z$$

or

$$\rho\,x; y; z$$

to mean that "x, y, z are in the relation ρ".

Finally, if τ is a quaternary relation and x, y, z, t are objects of any kind, we write

$$\tau\,x, y, z, t$$

or

$$\tau\,x; y; z; t$$

to mean that "x, y, z, t are in the relation τ".

In this section we introduce four fundamental relations: *Rqual*, *Rrelb*, *Rrelt*, *Rid*. The relation *Rqual* is a binary relation that connects *qualities* with *elements enjoying them.* Namely:

Axiom 1.3 – *Rqual is a binary relation. Given objects x, y, if*

$$Rqual\,x, y$$

holds, then x is a quality (i.e. x enjoys Qqual). Moreover, if q is a quality and x is any object, then $Rqual\,q, x$ holds if and only if $q\,x$ (i.e. x enjoys the quality q).

The relation *Rrelb* is a ternary relation, and it connects *binary relations* with *objects which are in these relations.* In other words:

Axiom 1.4 – *Rrelb is a ternary relation. Given objects* x, y, z, *if*

$$Rrelb\, x, y, z$$

holds, then x *is a binary relation (i.e.* x *enjoys Qrelb). Moreover, if* r *is a binary relation and* x, y *are objects of any kind, then* $Rrelb\, r, x, y$ *holds if and only if* $r\, x, y$ *(i.e.* x, y *are in the relation* r*).*

The relation *Rrelt* is a quaternary relation that connects *ternary relations* with *objects which are in these relations.* In other words:

Axiom 1.5 – *Rrelt is a quaternary relation. Given objects* x, y, z, t, *if*

$$Rrelt\, x, y, z, t$$

holds, then x *is a ternary relation (i.e.* x *enjoys Qrelt). Moreover, if* ρ *is a ternary relation and* x, y, z *are objects of any kind, then* $Rrelt\, \rho, x, y, z$ *holds if and only if* $\rho\, x, y, z$ *(i.e.* x, y, z *are in the relation* ρ*).*

Finally, *Rid* is a binary relation representing *identity.* In other words:

Axiom 1.6 – *Rid is a biunique relation. In order to have*

$$Rid\, x, y$$

it is necessary and sufficient that x *and* y *be the same object. In other words, every object is in the relation Rid with itself and only with itself.*

The relation *Rid* expresses the identity between objects of *any kind.* Hence we shall write usually $x = y$ instead of *Rid* x, y.

The relation *Rid* is the simplest instance of a ralation enjoing both qualities *Qrun* and *Qrbiun*. Concerning the quality *Qrun*, we state the following axiom, which expresses the idea of *univocity* applied to *binary, ternary* and *quaternary relations*:

Axiom 1.7 – *Qrun is a quality.*
1) if Qrun x*, then Qrel* x*;*
2) if Qrelb r; *Qrun* r; $r\ x, y$; $r\ x, z$ *, then* $y = z$*;*
3) if Qrelt ρ; *Qrun* ρ; $\rho\ x, y, z$; $\rho\ x, y, t$ *, then* $z = t$*;*
4) if Qrelq τ; *Qrun* τ $\tau\ x, y, z, t$; $\tau\ x, y, z, u$ *, then* $t = u$*.*

Finally *Qrbiun* is the quality of being a *binary biunique relation*, and biuniqueness is expressed by the axiom:

Axiom 1.8 – *Qrbiun is a quality.*
1) if Qrbiun x*, then Qrelb* x*, Qrun* x*;*
2) if Qrbiun x; $r\ y, x$; $r\ z, x$ *, then* $y = z$*.*

Operations, collections, sets and natural numbers

In this section we give some simple examples of "engrafting" mathematical notions into the "trunk" of the fundamental axioms concerning qualities and relations. Of course, each of these engraftings should be considered only a first "budd", from which whole "branches" of mathematics can develop in various ways. In this section we introduce the notions of operation, collection, set and natural number. We give only a few basic descriptive axioms, so as to leave open the possibility of developing the respective theories in different directions.

We introduce first the notion of *operation* by means of three qualities: *Qop*, the quality of being an operation; *Qops*, the quality of being a *simple operation*; *Qopb*, the quality of being a *binary operation*, and by means of two relations *Rops* and *Ropb*, which describe the way simple and binary operations operate. These objects satisfy the following axioms:

Axiom 2.1 – *Qops, Qopb, and Qop are qualities. Rops is a ternary univocal relation, and Ropb is a quaternary univocal relation.*
1) If either Qops x or Qopb x, then Qop x. (In other words, Qops and Qopb are specializations of the generic quality Qop.)
2) for all x, y, z*, if Rops* x, y, z*, then Qops* x*;*
3) if Ropb x, y, z, t*, then Qopb* x*.*

Whenever f is a simple operation, we often write $y = fx$ instead of *Rops* f, x, y, and we say that f *trasforms* x into y, or that f *maps* x onto y, or else that f *operates* on x giving the *result* y. Similarly, whenever φ is a binary operation, we often write $z = \varphi\ x, y$ instead of *Ropb* φ, x, y, z. Notice that *univocity* of the relations *Rops* and *Ropb* yields directly "functionality" of simple and binary operations, i.e. the fact that, if f is a simple operation operating on an object x, the result $y = fx$ is uniquely determined. Similarly, if φ is a binary operation operating on objects x, y, the result $z = \varphi\ x, y$ is uniquely determined.

After operations, we engraft another kind of objects, often considered in mathematics, namely *collections* and *sets*, which are particular collections, widely used in modern mathematics. To this aim, we introduce the quality *Qcoll*, i.e. the quality of being a *collection*, the relation *Rcoll*, the relation of *membership to collections*, the quality *Qins* of being a *set* and the relation *Rins* of *membership to sets*. These objects satisfy the following axioms:

Axiom 2.2 – *Qcoll and Qins are qualities. Rcoll and Rins are binary relations.*
1) Qins x implies Qcoll x;
2) if Rcoll x, y*, then x is a collection, i.e. Qcoll x;*
3) Rins x, y *if and only if Qins x, Rcoll* x, y*.*

Following the common usage, we write $y \in x$ instead of *Rcoll* x, y, and we say that y *belongs* to x, or that y is an *element* of x. The clause 3 says essentially that *Rins* is the restriction of *Rcoll* to sets.

We introduce also the relation *Rincl*, the relation of *inclusion*, satisfying the axiom:

Axiom 2.3 – *Rincl is a binary relation.*
1) If A, B *are collections, then Rincl* A, B *if and only if every element of A is also element of B;*
2) if Qins E, Qcoll X, and Rincl X, E*, then Qins X.*

Following the common usage, when A, B are collections and *Rincl* A, B holds, we write $A \subseteq B$ and we say that A is *included* in B or that A is a *part* of B. The clause 2) says that any collection included in a set is itself a set.

We can now state the fundamental axiom of the theory of collections, namely the *axiom of extensionality*:

Axiom 2.4 – *If* A, B *are collections,* $A \subseteq B, B \subseteq A$*, then* $A = B$*.*

In order to provide first instances of collections and sets we introduce now the *universal collection* V, the binary operation *Compl* (*relative complement*) and the simple operation *Csing* (generator of *singular collections* or *singletons*).

Axiom 2.5 – *V is a collection, Csing is a simple operation, Compl is a binary operation.*
1) For any object x it holds $x \in V$.
2) For any object x there exists $Csing\ x$ and it is a set whose unique element is x.
3) If A, B are collections, then $Compl\ A, B$ exists, and it is a collection whose elements are all and only those elements of A that are not elements of B.

We denote by $\{x\}$ the set $Csing\ x$, and we call it the *singleton* of x. We denote by $A \backslash B$ the collection $Compl\ A, B$, and we call it the *relative complement* of B w.r.t. A, or the *difference* between A and B.

Notice that it follows from the axioms that there exists an *empty collection* $\emptyset$, which is a set, and that there exist the *intersection* $A \cap B = A \setminus (A \setminus B)$ and the *union* $A \cup B = V \setminus ((V \setminus A) \cap (V \setminus B))$ of two collections A, B. In order to build up sets starting from singletons, we also postulate:

Axiom 2.6 – *If A, B are sets, then also $A \cup B$ is a set.*

So, given objects $x, y, z, t, \ldots$ and starting from their singletons, we get the *doubleton* $\{x, y\} = \{x\} \cup \{y\}$, the *tripleton* $\{x, y, z\} = \{x, y\} \cup \{z\}$, the *quartet* $\{x, y, z, t\} = \{x, y, z\} \cup \{t\}$, etc.

As previously remarked, we give here only the first "descriptive" axioms on operations and collections, so as to leave completely free space to developments of these notions by means of axioms specifying other ways of "costructing" collections, sets and operations (see, e.g. [15,18,19]).

Notice that we have carefully avoided the introduction of an axiom of extensionality for operations: in fact we intend to embody into the notion of operation the ideas of "construction", "manufacturing", "computing procedure", even giving to this idea an extremely broad intension, which in many cases, like those considered in the axiom 2.5, is very far from "effectiveness". So we will not exclude the possibility that two operations remain *different*, notwithstanding the fact that they operate on the *same objects* and they give always the *same results* (see [19]).

We conclude this section by introducing the natural numbers, i.e. the numbers 0, 1, 2, 3, 4, ..., by means of the quality *Qnnat*, of being a *natural number* and of the biunique relation *Rnsuc*, which connects each natural number with its *immediate successor*. These objects are ruled by the following axioms:

Axiom 2.7 – *Qnnat is a quality and Rnsuc is a biunique relation.*
1) $Rnsuc\ x, y$ implies $Qnnat\ x$, $Qnnat\ y$.
2) There exists a unique z such that $Qnnat\ z$ and for no x $Rnsuc\ x, z$.
3) If $Qnnat\, x$, then there exists y such that $Rnsuc\ x, y$.

Given a natural number x, the unique natural number y such that $Rnsuc\, x, y$ is called the *successor* of x. The axiom 2.7 suffices for characterizing the natural numbers 0 (the unique z of clause 2), 1 (the successor of 0), 2 (the successor of 1), etc.

The arithmetical theory provided by the axiom 2.7 is very weak. However it provides the possibility of defining *infinitely many* natural numbers by means of the relation *Rnsuc*. In case only a few natural numbers are needed, e.g. only

$0, 1, 2, 3, 4, 5, 6, 7$, one can replace the clause 3 in axiom 2.7 by some particular instances, for instance one can assume the axioms:

2.7.3.1) There exists the natural number 1 *such that Rnsuc* 0; 1*;*
2.7.3.2) There exists the natural number 2 *such that Rnsuc* 1; 2*;*
2.7.3.3) There exists the natural number 3 *such that Rnsuc* 2; 3*;*
2.7.3.4) There exists the natural number 4 *such that Rnsuc* 3; 4*;*
2.7.3.5) There exists the natural number 5 *such that Rnsuc* 4; 5*;*
2.7.3.6) There exists the natural number 6 *such that Rnsuc* 5; 6*;*
2.7.3.7) There exists the natural number 7 *such that Rnsuc* 6; 7*.*

The development of various strong theories of (standard and nonstandard) arithmetic might depend on the introduction of operations (beginning with the four operations of the primary scool), relations (natural ordering, divisibility, etc.), collections (the collection **N** of all natural numbers, the collection **P** of all primes, etc.), various forms of the induction principle, etc.

Correlations, functions and systems

In this section we consider another kind of fundamental objects of mathematics, namely the correlations, together with the special cases of functional correlations, systems and functions.

In the usual treatments, systems and functions are often identified with their graphs, obtained by means of the so called "Kuratowski pairs", i.e. sets of the type $\{\{x\}, \{x, y\}\}$. We prefer instead to consider correlations as a kind of objects quite separate from the kind of collections, so as to leave open way to various theories concerning the relations between collections and correlations: see [18] for a first example of such theories. In this section we restrict ourselves to ascertain some similarities between the theory of collections and that of correlations. In particular we exploit the possibility of extending to correlations some axioms concerning the relation *Rincl* and the operation *Compl* already considered in the previous section.

We also remark that the notion of operation is quite separate from those of functional correlation and of function, notwithstanding the fact that all of them are characterized by univocity axioms: operations do not satisfy any axiom of extensionality, insofar they bring about the intuitions of manufacturing, computing, constructing, which are essentially non-extensional. The idea of correlation, instead, is inspired to a mere "inspection of input-output tables". The attitude of an engineer who has to organize the work of a factory corresponds, in some sense, to the concept of operation; the concept of correlation is closer to the attitude of a warehouse-keeper, who has simply to register ingoing materials and outgoing products.

A first step in engrafting correlations is done by introducing the quality *Qcorr* of being a *correlation* and the relation *Rcorr* that describes the action of correlations. About these objects we state the axiom:

Axiom 3.1 – *Qcorr is a quality. Rcorr is a ternary relation. If Rcorr* x, y, z*, then Qcorr* x*.*

Given a correlation C, instead of writing *Rcorr* C, x, y we can say that x is an *index* of C, that y is a *value* of C, and that the index x and the value y are *correlated* by C.

Having engrafted correlations, we can pass to functional correlations, systems, and functions by introducing the qualities $Qcorfun$, of being a *functional correlation*, $Qsys$ of being a *system*, and $Qfun$ of being a *function*. These qualities satisfy the axiom:

Axiom 3.2 – *$Qcorfun, Qsys$ and $Qfun$ are qualities.*
1) If $Qcorfun\ x$ or $Qsys\ x$, then $Qcorr\ x$.
2) $Qfun\ x$ if and only if x enjoys simultaneously $Qsys$ and $Qcorfun$.

We introduce also the relations $Rcorfun, Rsys, Rfun$, the restrictions of *Rcorr* to functional correlations, systems, and functions respectively. This fact is expressed by the following axiom:

Axiom 3.3 – *$Rcorfun, Rsys, Rfun$ are ternary relations.*
1) $Rcorfun\ F, x, y$ if and only if $Rcorr\ F, x, y$ and $Qcorfun\ F$.
2) $Rsys\ S, x, y$ if and only if $Rcorr\ S, x, y$ and $Qsys\ S$.
3) $Rfun\ f, x, y$ if and only if $Rcorr\ f, x, y$ and $Qfun\ f$.

We can now give the axiom of *univocity* for functional correlations and functions:

Axiom 3.4 – *$Rcorfun$ and $Rfun$ are univocal ternary relations.*
Let F be a correlation: then $Qcorfun\ F$ holds (i.e. F is a functional correlation) if and only if whenever x, y, z satisfy simultaneously the conditions $Rcorr\ F, x, y$ and $Rcorr\ F, x, z$, then $y = z$.

The inclusion relation *Rincl*, already introduced in the preceding section, may concern, besides collections, also other objects like correlations, for which we give an axiom analogous to the axiom 2.3:

Axiom 3.5 – *Let F, G be correlations: then $Rincl\ F, G$ if and only if, for all x, y, $Rcorr\ F, x, y$ implies $Rcorr\ G, x, y$ (in other words, $Rincl\ F, G$ holds if and only if indices and values correlated by F are also correlated by G). Moreover, if s is a system, t is a correlation and $Rincl\ t, s$ holds, then also t is a system.*

Also in the case of correlations we use a notation similar to that used for collections, i.e. $F \subseteq G$ or $G \supseteq F$ instead of $Rincl\ F, G$, and we say that the correlation G includes the correlation F, or that F is a part of the correlation G. We give the *axiom of extensionality* for correlations in the following way:

Axiom 3.6 – *If F and G are correlations and both $F \subseteq G, G \subseteq F$ hold, then $F = G$.*

We introduce now the relation *Rdom*, connecting a correlation with its *indices* (also called the "elements of its domain") and the relation *Rcod*, connecting a correlation with its *values* (also called the "elements of its codomain").

Axiom 3.7 – *Rdom and Rcod are binary relations. If F is a correlation, then one has:*
1) $Rdom\ F, x$ if and only if there exists y such that $Rcorr\ F, x, y$;
2) $Rcod\ F, y$ if and only if there exists x such that $Rcorr\ F, x, y$.

In this paper we do not postulate that for any correlation F there is a collection whose elements are all and only the indices of F, nor we postulate that there is one with all and only the values of F. On the other hand, we shall see in a moment that the relations *Rdom* and *Rcod* concern also other kinds of objects, in particular operations and relations.

However we want to ensure first the existence of a few correlations of particular relevance. To this aim we introduce the universal correlation V_2 and the operation $Fsing$, generator of *singular functions*, and we extend to correlations the operation $Compl$ previously introduced. We set the axioms:

Axiom 3.8 – *1) Fsing is a binary operation. For any given objects* a, b *there exists* $Fsing\ a, b = f$*, and* f *is a function such that* $Rfun\ f, x, y$ *holds if and only if* $x = a$ *and* $y = b$*.*
2) V_2 *is a correlation such that Rcorr* V_2, x, y *holds for all* x, y*.*

Axiom 3.9 – *If* F, G *are correlations, then* $Compl\ F, G = H$ *exists, and* H *is a correlation such that, for all* x, y*, Rcorr* H, x, y *if and only if Rcorr* F, x, y*, but not Rcorr* G, x, y*.*

Inspired by the algebraic notation of substitutions, we denote $\binom{a}{b}$ the singular function $Fsing\ a, b$. Moreover we extend to correlations the notation of collections: the *difference* $F \setminus G = Compl\ F, G$; the *intersection* $F \cap G = F \setminus (F \setminus G)$; the *union* $F \cup G = V_2 \setminus ((V_2 \setminus F) \cap (V_2 \setminus G))$.

In this notation, the conditions $Rcorr\ F, x, y$ and $F \supseteq \binom{x}{y}$ are equivalent for any correlation F.

We can now introduce the relation $Rgraf$, connecting a correlation to the *singular functions* included in it (also called at times "elements of its graph"). More precisely, we state the axiom:

Axiom 3.10 – $Rgraf$ *is a binary relation.*
1) If F *is a correlation and* $Rgraf\ F, z$ *holds, then* z *is a singular function;*
2) for all x, y *one has* $Rgraf\ F, \binom{x}{y}$ *if and only if* $F \supseteq \binom{x}{y}$*.*

We remark that the relations $Rdom, Rcod, Rgraf$ involve, besides correlations, also relations, on wich we give the following axiom:

Axiom 3.11 – *1) If* r *is a binary relation and* x *is any object, one has Rdom* r, x *if and only if there exists* y *such that* r, x, y*; one has Rcod* r, y *if and only if there exists* x *such that* r, x, y*; one has* $Rgraf\ r, w$ *if and only if there exist* x, y *such that* $w = \binom{x}{y}$ *and* $r\ x, y$*.*
2) If ρ *is a ternary relation, one has Rdom* ρ, w *if and only if there exist* x, y *such that* $w = \binom{x}{y}$ *and there exists* z *such that* $\rho\ x, y, z$*; one has Rcod* ρ, z *if and only if there exist* x, y *such that* $\rho\ x, y, z$*; one has* $Rgraf\ \rho, z$ *if and only if there exist* a, b, c *such that* $z = \binom{\binom{a}{b}}{c}$ *and* $\rho\ a, b, c$*.*
3) If τ *is a quaternary relation, then one has Rdom* τ, w *if and only if there exist* a, b, c *such that* $w = \binom{\binom{a}{b}}{c}$ *and there exists* d *such that* $\tau\ a, b, c, d$*; one has Rcod* τ, d *if and only if there exist* a, b, c *such that* $\tau\ a, b, c, d$*; one has* $Rgraf\ \tau, z$ *if and only if there exist* a, b, c, d *such that* $z = \binom{\binom{\binom{a}{b}}{c}}{d}$ *and* $\tau\ a, b, c, d$*.*

A similar axiom could be given also for simple and binary operations. Notice that the relation $Rgraf$ connects quaternary relations and singular functions, which in turn have been introduced by appealing to ternary relations and binary operations; we close in this way a sort of cicle within the first three sections of this note, and an interesting example of *selfreference*, or better *mutual reference* comes off. Much harder problems concerning possible and impossible selfreferences arise from reflexions upon the next section, where we deal with propositions and predicates.

We conclude this section by pointing out two more interesting operations acting on correlations, the operation of inversion and the operation of composition; we give also some instances of interesting systems that can be constructed by union of correlations, starting from singular functions. We introduce the operations Inv (inversion) and $Comp$ (composition), fulfilling the axioms:

Axiom 3.12 – *Inv is a simple operation.*
1) If F is a correlation, then $Inv\ F = G$ exists, and G is a correlation such that $Rcorr\ G, x, y$ if and only if $Rcorr\ F, y, x$.
2) If s is a system, then $Inv\ s$ is a system.

Axiom 3.13 – *$Comp$ is a binary operation.*
1) If F, G are correlations, then $Comp\ F, G = H$ exists, and H is a correlation; moreover one has $Rcorr\ H, x, y$ if and only if there exists z such that $Rcorr\ G, x, z$ and $Rcorr\ F, z, y$.
2) If s, t are systems, then $Comp\ s, t$ is a system.

In order to shortly indicate inverse and composite correlations we adopt the notation $F^{-1} = Inv\ F$, $F \circ G = Comp\ F, G$.

The axioms we introduced up to now allow for manufacturing "by hand" those "finite" systems that one actually want tu use. For instance, given two singular operations $\binom{a}{b}, \binom{c}{d}$, one gets by union the system $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$; similarly, given a third singular function $\binom{e}{f}$ we get the system $\begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix}$. In particular, for any integer n defined according to axiom 2.7, or according to the axioms 2.7.3.1 ,..., 2.7.3.7, one can consider the n-tuples as functions defined at the numbers $1, 2, \ldots, n$. In particular for avery object a one has the "1-tuple" $\binom{1}{a}$, given objects a, b one has the 2-tuple, or ordered pair $\begin{pmatrix} 1 & 2 \\ a & b \end{pmatrix}$, which will be denoted by the symbol (a, b), given objects a, b, c one has the triple $\begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}$, denoted by (a, b, c), and given objects a, b, c, d one has the 4-tuple $\begin{pmatrix} 1 & 2 & 3 & 4 \\ a & b & c & d \end{pmatrix}$, denoted by (a, b, c, d), etc.

Propositions and predicates

In this section we introduce some very general ideas about propositions and predicates. The axioms we are introducing are intentionally very weak, and they deal essentially with "classical" propositions and predicates, so as to leave widest freedom to subsequent development of other logics, different from classical logic, as well as to an investigation of the difficult problems arising in passing from the study of "restricted" models, whose domain is a set (as usual in model theory), to the study of "universal" models of different kinds of logics, whose domain is the universal collection V.

Therefore we begin by introducing the "most general" notion of proposition by means of the quality *Qgprop*: so we write $Qgprop\ x$ to intend that x is a *proposition*, without any further specification. Among all propositions there are in particular the "classical" propositions, which obey the rules of classical propositional calculus. These propositions are characterized by the quality *Qclp*: so $Qclp\ x$ means that x is a *classical proposition*. We can state the axiom:

Axiom 4.1 – *$Qgprop, Qclp$ are qualities. If $Qclp\ p$, then $Qgprop\ p$.*

The traditional logical operations *Et* (*conjunction*), Vel (*disjunction* or *alternative*), Non (*negation*) act on propositions:

Axiom 4.2 – *Et and* Vel *are binary operations.* Non *is a simple operation. If* p, q *are classical propositions, then Et* p, q*,* Vel p, q*,* Non p *exist and are classical propositions.*

Following the common usage, we denote by $p\&q$, $p \vee q$, $\neg p$ the propositions Et p, q, Vel p, q, and Non p.

We consider also the quality *Qtver*, of being "absolutely and totally" true, and so we write *Qtver* x to affirm that x is *true*, and we write simultaneously *Qgprop* p, *Qtver* p to affirm that p is a *true proposition.* The following axiom connects the classical propositions to the quality *Qtver*, and expresses, *inter alia*, the classical principles of *non contradiction* and of the *excluded middle*:

Axiom 4.3 – *Qtver is a quality. If* p, q *are classical propositions, then:*
1) p *enjoys Qtver if and only if* $\neg$ p *does not enjoy Qtver.*
2) $p\&q$ *enjoys Qtver if and only if both* p *and* q *enjoy Qtver.*
3) $p \vee q$ *enjoys Qtver if and only if at least one of* p *and* q *enjoys Qtver.*

When $\neg p$ enjoys the quality *Qtver*, we say also that p is *false*.

We introduce also the relation *Rseq* of *logical consequence* and the relation *Rleq* of *logical equivalence* between propositions. We give for these relations only two very general axioms, restricting ourselves, as customary, to the case of classical propositions:

Axiom 4.4 – *Let* p, q, r *be classical propositions. If Rseq* p, q *and Qtver* p*, then Qtver* q*. Moreover one has:*
1) Rleq p, q *if and only if Rseq* p, q*, Rseq* q, p*.*
2) Rseq p, p*.*
3) If Rseq p, q *and Rseq* q, r*, then Rseq* p, r*.*

Axiom 4.5 – *Let* p, q *be classical propositions. Then one has:*
1) Rseq $p\&q, p$ *and Rseq* $p\&q, q\&p$*.*
2) Rseq $p, p \vee q$ *and Rseq* $q \vee p, p \vee q$*.*
3) Rleq $p, \neg\neg p$*.*

Having thus introduced propositions, we can resume the classical notion of predicate "*à la* Frege" (see [20]), i.e. as an operation taking on propositions as values. Of course, many different introductions are possible (see, e.g., [13,15,25]). Here we introduce the quality *Qopreds* of being a *simple predicative operation* and the quality *Qclops* of being a *simple classical predicative operation.* The following axiom holds:

Axiom 4.6 – *Qopreds and Qclops are qualities.*
1) If Qclops x*, then Qopreds* x*.*
2) If Qopreds x*, then Qops* x*.*
3) If Qopreds x *and Rops* x, y, z*, then Qgprop* z*.*
4) If Qclops x *and Rops* x, y, z*, then Qclp* z*.*

The connections between the objects studied in the previous three sections and the propositions describing them are established by the operation *Gops*, the generator of *simple predicative operations.* This operation fulfils the following axioms:

Axiom 4.7 – *Gops is a simple operation. If* τ *is a quaternary relation, then:*

1) Gops τ is a simple operation;
2) for every object x, $(Gops\ \tau)x$ is a simple operation;
3) for all x, y, $((Gops\ \tau)x)y$ is a simple operation;
4) for all x, y, z, $((Gops\ \tau)x)y)z$ is a simple predicative operation;
5) for all x, y, z, t, $(((Gops\ \tau)x)y)z)t$ is a proposition;
6) for all x, y, z, t, the proposition $(((Gops\ \tau)x)y)z)t$ enjoys Qtver if and only if $\tau\ x, y, z, t$ holds (i.e. if and only if x, y, z, t are in the relation τ).

The clause 6 of the axiom above justifies the shortened notation

$$\text{“}\tau\ x, y, z, t\text{”} = ((((Gops\ \tau)x)y)z)t.$$

After defining the behaviour of the operation *Gops* at quaternary relations it is an easy task to pass to other kinds of objects introduced in the previous sections, by operating in the following way:

Axiom 4.8 – *1) If ρ is a ternary relation, then Gops $\rho = (Gops\ Rrelt)\rho$.*
2) If r is a binary relation, then Gops $r = (Gops\ Rrelb)r$.
3) If q is a quality, then Gops $q = (Gops\ Rqual)q$.
4) If f is a simple operation, then Gops $f = (Gops\ Rops)f$.
5) If φ is a binary operation, then Gops $\varphi = (Gops\ Ropb)\varphi$.
6) If C is a collection, then Gops $C = (Gops\ Rcoll)C$.
7) If F is a correlation, then Gops $F = (Gops\ Rcorr)F$.

The “elementary” propositions generated by predicates obtained by means of the operation *Gops* are statements concerning those qualities or relations to which *Gops* has been applied. Hence we extend the quoted notation in the natural way:
if q is a quality, then “qx” stands for $(Gops\ q)x$;
if r is a binary relation,, then “$r\ x, y$” stands for $((Gops\ r)x)y$;
if ρ is a ternary relation, then “$\rho\ x, y, z$” stands for $(((Gops\ \rho)x)y)z)$;
if C is a collection, then “$x \in C$” stands for $(Gops\ C)x$;
finally “$x = y$” stands for $((Gops\ Rid)x)y$.

We can now introduce the *existential quantifier* and the *universal quantifier* by means of the operations *Exist* and *Univ*, ruled by the axiom:

Axiom 4.9 – *Univ and Exist are simple operations.*
1) If p enjoys Qclops, then both Univ p and Exist p exist and enjoy Qclp.
2) The proposition Univ p is true if and only if, whenever the proposition px exists, it is true.
3) The proposition Exist p is true if and only if, for at least an object x, the proposition px exists and it is true.

We shall use the more common notation $\forall p$, $\forall x.px$, $\forall y.py$, etc. instead of *Univ* p, and similarly $\exists\, p, \exists x.px$, $\exists y.py$, etc. instead of *Exist* p: of course, these expressions do not refer to any specified objects x, y, etc.

Having introduced operations and predicates, and in particular the “elementary” predicates defined through *Gops*, two problems arise, which might be analyzed in several directions. The first problem regards the study of propositions and predicates that can be built up by means of various manipulations of operations, starting from *Gops* and from the logical operations $Et, Vel, Non, Exist, Univ$. Solutions to a similar problem have been given in [13,15,25], but obviously there are many different, equally interesting possible solutions. More difficult is the second problem, which leads us into the neighborhood of all antinomies and paradoxes: namely that of discerning which, among the propositions and predicates built up by means of *Gops*, could be *classical* propositions and predicates. Some negative results, essentially inspired by the Liar's Paradox, or by Tarski's theorem, have been proved in similar contexts, see [13,15]. The search for the strongest positive axioms that do not lead to contradiction is still open.

Concluding remarks

Besides the objects introduced in the last three sections above, one could of course consider, within the framework of the most general axioms on qualities and relations, other objects like *variables* (see [5,11]), *categories*, and possibly resume into consideration the *metaqualities*, introduced in [13] as intermediate objects between the level of qualities and relations, "premathematical" objects, and the level of "mathematical" objects like operations, numbers, correlations and collections. Each theory can be developed in several directions, and so we hope that many scholars with different cultural education and various interests take part in these developments, retaining the greatest freedom and independence, but also sharing great willingness of a sincere comparison of ideas. The section concerning propositions and predicates is probably the one that will face more problems and greater difficulties, but it might also give rise to the newest, most interesting ideas. Indeed it is in this field that one reaches the deepest common roots of mathematics, logic and informatics, and one gets close to ancient and modern paradoxes and antinomies.

The situation of the scholar who wants to study in depth these topics resembles that of a mountaineer who advances, on very narrow ridges, surrounded by deep ravines, proceeding towards high, beautiful peaks. In fact, one treads on the dangerous ground of "selfreference" and of "mutual reference" (cfr. [14,24]): these topics implicate treacherous problems, but the ultimate significance of the different forms of human knowledge stands out therefrom. In order to get a clear understanding of these questions, a frank and open discussion among scholars of diverse disciplines is necessary: a restricted debate among specialists is unappropriate. Notice that various forms of selfreference (or of mutual reference) are met not only in mathematics, logic and computer science, but also in many other branches of human knowledge. One may quote dictionaries that explain a word through other words, and that contain also the word "dictionary", grammars written according to the grammar rules, laws that regulate the lawmaking activity, economic doctrines that, if successful, describe economic phenomena that depend on the actions of agents deeply influenced by these doctrines. Other examples are the history of historiography, the painter who paints himself while painting, the theatre on the stage, the novelist or the poet who talks about the writing of a novel or about the creation of a poem, the biologist investigating the relations between eyes and brain on the grounds of observations made by her own eyes, etc.

Countless other cases of selfreference and mutual reference could be mentioned, and perhaps the respective similarities and differences could be the subject matter of a frank interdisciplinary comparison of ideas. In this spirit of frank comparison of ideas have to be intended also the statements of this paper, which are "proposed" (not "imposed") as "axioms", i.e. statements not deduced from a system of previous statements, but chosen as a possible starting point for further developments of various theories. Neither are these axioms "deduced" from the history of mathematics, from the philosophy of science, from logic, etc. On the contrary, they can be better understood by assuming an attitude as naïve as possible, and by keeping to the most common meanings that every day language assigns to the words quality and relation. Only after such a first reading of this paper it is appropriate a critical second reading, where everybody can obviously bring about their own experiences and knowledge of mathematics, logic, computer science, history, philosophy, physics, economics, etc.: these esperiences could be very important in setting forth deeper and better justified critical valuations, and hopefully in suggesting supplementary or alternative "axioms", which better represent and clarify the involved intuitive notions.

We do not expect, nor even want, that our proposal be unconditionally accepted: we believe that a vaste work of critical reflexion and friendly discussions between scholars of diverse education and attitudes is still needed, in order to attain a good axiomatic system (or some good axiomatic systems), suitable for substantially improving upon the present situation. On the other hand, we think that it is worth while tu pursue this goal, because neither formalism, nor set theoretic reductionism, along with any other form of reductionism, seem to offer a suitable perspective for a real understanding of many problems that the culture of our time has to face.

Any investigation of the fundamental axioms of mathematics, logic and computer science, as well as of various experimental, human and philosophical sciences needs, among other things, to overcome a too restricted vision of the different specialities, and requires a broader idea of mathematical and scientific rigor. Mathematical rigor is not only carefulness of the proofs, but also engagement in exposing, in the most clear and understandable way, the problems one would want to solve, the theorems one would want to prove, the conjectures one would want to verify or refute. We think that scientific rigor ultimately consists in clearly and frankly exposing their own certainties and doubts, which problems one believes to have solved and which one would like to solve or see solved, while avoiding those confuse, obscure, uselessly complicated talks that end up in annoying even the most favourably disposed listener.

Summing up, we may conclude that any consideration on method, rigor, and meaning of science leads us in the end to the ancient intuitions of the sapiential value of humility, of "conviviality" (which keeps together sharing of knowledge, friendship, search of mutual understanding), and of trust in Wisdom, which meets all those who love and seek it.

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